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ON CLAWS BELONGING TO EVERY TOURNAMENT

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A directed graph is said to be n-unavoidable if it is contained as a subgraph by every tournament on n vertices. A number of theorems have been proven showing that certain graphs are n-unavoidable, the first being Rédei's result that every tournament has a Hamiltonian path. M. Saks and V. Sós gave more examples in [6] and also a conjecture that states: Every directed claw on n vertices such that the outdegree of the root is at most $[n/2]$ is n-unavoidable. Here a claw is a rooted tree obtained by identifying the roots of a set of directed paths. We give a counterexample to this conjecture and prove the following result: *any claw of rootdegree* $\leq n/4$ *is n-unavoidable.*

1. Introduction

It is well known that every tournament contains a Hamiltonian path and thus the path of size n is n-unavoidable in any tournament of size n. A directed graph G is said to be n-unavoidable if every tournament contains it as a subgraph and we denote the set of all n-unavoidable n-digraphs by $U(n)$. The problem of identifying classes of n-unavoidable digraphs has been considered by several researchers ([1], [2], [3], [4], [5], [6]). Here, we use size to denote the number of vertices.

Let $L = (l_1, l_2, \ldots, l_r)$ be a sequence of nonnegative integers. A *claw* $C(L)$ is a rooted directed tree obtained by identifying the roots of dipaths of sizes $l_1 + 1$, $l_2 + 1$, $\ldots$, $l_r + 1$. $C(L)$ has $1 + \sum_{i=1}^{r} l_i$ vertices. A *k-claw* is a claw in which all l_i's are k, except possibly one of them is less than k. In [6], M. Saks and V. Sós considered the question which claws on n vertices are n-unavoidable. Since there exists tournament on n vertices such that the outdegree of every vertex is at most $[n/2]$, a necessary condition for a claw to be n-unavoidable is for the root to have outdegree at most $[n/2]$. They conjectured that this condition is also sufficient. In particular they conjectured that the 2-claw of size n is n-unavoidable and showed that this would imply the general conjecture. In this note we show that the 2-claw of size n is not in $U(n)$ for infinitely many n. In other words, we shall prove

Theorem 1. *There exist infinitely many n such that the 2-claw on n vertices is not n-unavoidable.*

On the other hand we obtain the following positive result:

Theorem 2. *Any claw of root degree $\leq n/4$ is n-unavoidable.*

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An interesting question that remains is: What is the maximum value c such that every claw of root degree $\leq cn$ is n-unavoidable. We propose the following

Problem: What is the value of c?

We say a vertex is a *good vertex* of a tournament T if it is the root of a 2-claw of size $n = |T|$. For vertices x and w, Let $I(x, w)$ be the set of vertices y such that $x \to y$ and $y \to w$ in T and for a subset W, let $I(x, W) = \bigcup_{w \in W} I(x, w)$. Two consequences of the definitions are:

Proposition 1. *If x is a good vertex of T, then $|I(x, W)| \geq |W|$, where $W \subseteq \{y \in T | y \to x\}$.*

Proposition 2. *If x is a good vertex of T, then there is a path of length at most 2 from x to any other vertex w.*

2. Proofs of Theorems

Proof of Theorem 1. We construct a tournament T in the following way. First, we construct a tournament H such that there is only one good vertex in H and having other properties. Then we use H to get a larger tournament T such that there is no good vertex in T at all.

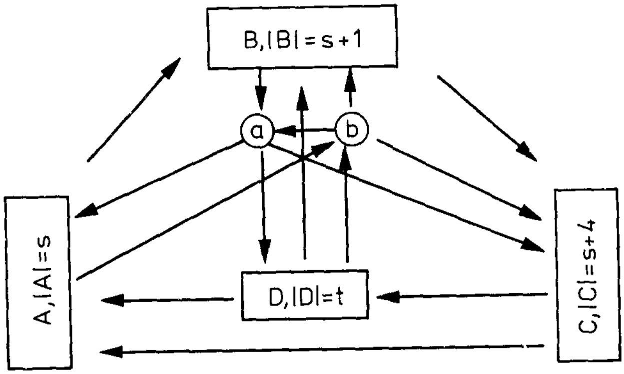

Fig. 1

For two vertex sets A and B, let (A, B) denote the set of arcs directed from A to B. Now let s be a positive integer, A, B and C be arbitrary tournaments of size $s, s+1$ and $s+4$ respectively. Let D be an arbitrary tournaments of size $t \geq 7$. We construct H as follows. $V(H) = V(A) \cup V(B) \cup V(C) \cup V(D) \cup \{a, b\}$. The arcs of H consist of those belonging to A, B, C and D in addition to (A, B), (B, C), (C, A), (D, A), (D, B), (C, D), (a, A), (a, C), (a, D), (B, a), (b, a), (b, B), (b, C), (A, b) and (D, b) (see Figure 1). We have the following facts.

1 If $x \in A$, then x cannot be a good vertex since every path from x to D has length ≥ 3.

2. If $x \in B$, then x cannot be a good vertex since every path from x to b has length ≥ 3.

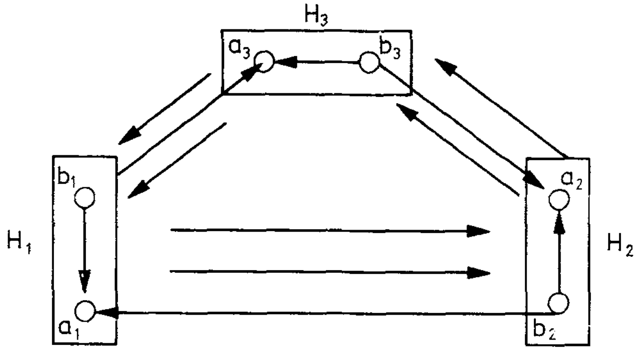

Fig. 2

3. If $x \in C$, then x cannot be a good vertex since every path from x to a has length ≥ 3.

4. If $x \in D$, then x cannot be a good vertex since $|I(x, C \cup \{a\})| = |B \cup \{b\}| = |B| + 1 = s + 2$ but $|C| + 1 = s + 5$.

5. b cannot be a good vertex since $|I(b, A \cup D)| = |C \cup \{a\}| = |C| + 1 = s + 5$ but $|A \cup D| = |A| + |D| = s + t \geq s + 7$.

Hence only a could be a good vertex of H (and indeed it is).

Next we construct the tournament T. Let H_1, H_2 and H_3 be copies of H. The set $A(T)$ of arcs of T consists of $A(H_1) \cup A(H_2) \cup A(H_3)$ together with $\{(H_i, H_{i+1}) - (a_i, b_{i+1})\} \cup (b_{i+1}, a_i)$, $1 \leq i \leq 3$, where the subscripts are taken mod 3 (see Figure 2). Now we can prove that there is no good vertex in T. It is enough to show that there is no good vertex in H_1 due to the symmetry of H_1, H_2 and H_3. For above T, we have the following results by using proposition 1 and 2.

6. a_1 cannot be a good vertex since $|I(a_1, H_3)| = |H_2 - \{b_2\}| < |H_3|$.

7. b_1 cannot be a good vertex since we have $I(b_1, A_1 \cup D_1) = C_1 \cup \{a_1, a_3\}$ and $|A_1 \cup D_1| = s + t \geq s + 7 > |C_1 \cup \{a_1, a_3\}| = s + 6$.

8. If x is in $A_1 \cup B_1 \cup C_1$ then x cannot be a good vertex as shown in 1, 2 and 3 above.

9. If x is in D_1 then x cannot be a good vertex since $|I(x, C_1 \cup \{a_1\})| = |B_1 \cup \{b_1\} \cup \{b_2\}| = s + 3 < s + 5 = |C_1 \cup \{a_1\}|$

From the facts above we see that there is no good vertex in T.

Thus we complete the proof of theorem 1.

Proof of Theorem 2. Recall that a claw of size n can be identified with a vector $L = (l_1, l_2, \ldots, l_k)$ where $l_1 \geq l_2 \geq \ldots \geq l_k$ and $l_1 + l_2 + \ldots + l_k = n - 1$, i.e. a partition of $n - 1$. The standard majorization order on the partitions of $n - 1$ given by

$$L = (l_1, l_2, \ldots, l_r) \geq K = (k_1, k_2, \ldots, k_s)$$

if

$$\sum_{i=1}^{j} l_i \geq \sum_{i=1}^{j} k_i$$

for each $j \geq 1$ defines an order on claws of size n. In [6], M. Saks and V. Sós noted the following

Lemma 1. *Let $C(L) \geq C(K)$ be both claws of order n. If $C(K)$ is in T, then so is $C(L)$. In particular, if $C(K)$ is in $U(n)$, then so is $C(L)$.*

It is easy to see that in this order the k-claw is less than or equal to any claw of root degree $\leq n/k$, so Theorem 2 follows from

Theorem 3. *The 4-claw of size n is in $U(n)$.*

So we turn to proof of Theorem 3. First, we list some notations. Let T be a tournament of size n. We use $V(T)$ and $A(T)$ to denote the vertex set and arc set of T, respectively. Let x be a vertex of T, the sets of out-neighbours and in-neighbours of x are denoted by $N_T^+(x)$ and $N_T^-(x)$ respectively. In other words, $N_T^+(x) = \{y|xy \in A(T)\}$, and $N_T^-(x) = \{y|yx \in A(T)\}$. Let $d_T^+(x) = |N_T^-(x)|$ and $d_T^-(x) = |N_T^-(x)|$. Now, we are ready to prove Theorem 3.

Let T be an arbitrary tournament of order n and x be a vertex with maximum outdegree in T, that is, $d_T^+(x) = \max\{d_T^+(y)|y \in V(T)\}$. Let $A = N_T^+(x)$, $B = N_T^-(x)$. Let $b = |B|$ and $a = |A|$, where $a \geq b$ by the choice of x. Let $G = (A, B; E)$ be the bipartite graph obtained from T with edge set $E = \{(u,v) \in A(T)|u \in A, v \in B\}$. Let M be a maximum matching of G. We have

Lemma 2. $|M| > b/2$

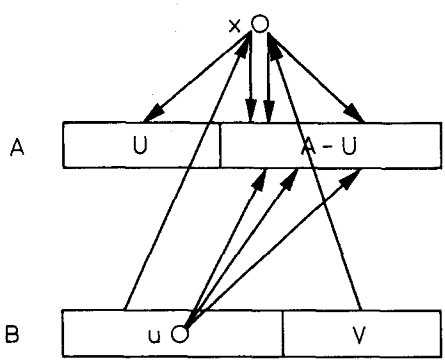

Fig. 3

Proof. Let $U \cup V$ be the minimum cover of G where $U \subseteq A$ and $V \subseteq B$, which means that every edge of G must meet $U \cup V$. By König's theorem only need to show $|U \cup V| > b/2$. Let T' be the subtournament of T obtained from T by restricting on $B - V$, and u be a vertex of T' with maximum out-degree in T', and this implies that $d_{T'}^+(u) \geq \frac{|B|-|V|-1}{2}$. Notice that $U \cup V$ is a cover of G, we must have $(u, A - U) \subseteq A(T)$, (see Figure 3), thus $(u, A - U \cup \{x\}) \subseteq A(T)$ since $ux \in A(T)$. By the choice of x, we have $d_T^+(x) \geq d_T^+(u)$. Thus,

$$a \geq d_T^+(u) \geq d_{T'}^+(u) + |(u, A - U \cup \{x\})|$$
$$\geq \frac{|B| - |V| - 1}{2} + |A| - |U| + 1 = \frac{b - |V| + 1}{2} + a - |U|$$

Hence

$$|U \cup V| = |U| + |V| \geq \frac{b+1+|V|}{2} > b/2$$

Next, let $X = A \cap V(M), Y = B \cap V(M), Y' = B - Y$ and G^* be the bipartite graph on $(X \cup Y, Y'; E^*)$, where $E^* = \{(u,v) \in A(T) | u \in X \cup Y, v \in Y'\}$. Let M^* be a maximum matching in G^*. We have

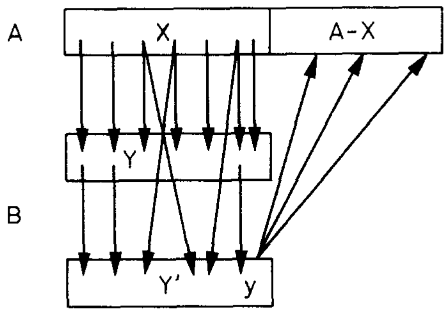

Fig. 4

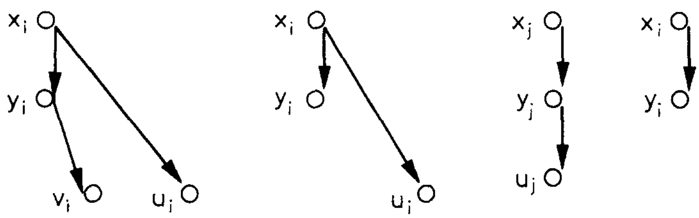

Fig. 5

Lemma 3. $|M^*| = |Y'|$

Proof. By Hall's matching theorem it is enough to show that in G^*, $d(y) > |Y'|$ for any $y \in Y'$. Since M is a maximum matching in G, we get $(y, A - X) \subseteq A(T)$ (see Figure 4). Note that $|(y, X \cup Y)| = |X \cup Y| - d(y)$, and $yx \in A(T)$, hence

$$a \geq d_T^+(y) \geq a - |X| + 1 + |X \cup Y| - d(y) = a + 1 + |Y| - d(y)$$

i.e.

$$d(y) \geq 1 + |Y| = 1 + |M| > 1 + b/2 > |Y'|.$$

Now we finish the proof of Theorem 3. Let $X = (x_1, x_2, \ldots, x_r)$ and $Y = (y_1, y_2, \ldots, y_r)$ where $x_i y_i \in M$. Let $I = \{i | x_i \in X \cap M^*\}$ and $J = \{j | y_j \in Y \cap M^*\}$. If $I \cap J = 0$, then we can get the 3-claw of size n in T. If $I \cap J \neq 0$, choose $i \in I \cap J$, i.e. $\exists u_i, v_i \in Y'$ such that $x_i u_i \in M^*$, $y_i v_i \in M^*$, then if $u_i v_i \in A(T)$, we have either $x \to x_i \to y_i \to u_i \to v_i$ or $x \to x_i \to u_i \to y_i \to v_i$ as paths in T. If $v_i u_i \in A(T)$ then $x \to x_i \to y_i \to v_i \to u_i$ is a path in T. We can deal with each $i \in I \cap J$ this way. If $i \in I - J$, and $x_i u_i \in M^*$, then either $x \to x_i \to y_i \to u_i$ or $x \to x_i \to u_i \to y_i$ is a path. If $j \in J - I$, and $y_j u_j \in M^*$, then $x \to x_j \to y_j \to u_j$ is a path. For those

$i \in \{1, 2, \ldots, r\} - I \cup J, x \to x_i \to y_i$ is a path and for $u \in A - X, x \to u$ is a path. Figures 4 and 5 can help us to understand the situation. Thus, if we let

$$\alpha = |I \cap J|$$
$$\beta = |I| + |J| - 2|I \cap J|$$
$$\gamma = |Y| - |Y'| + |I \cap J|$$

and

$$\delta = |A - X|.$$

Then we can construct a claw $C(L)$ in T with root x and

$$L = (\overbrace{4, 4, \ldots, 4}^{\alpha}, \overbrace{3, 3, \ldots, 3}^{\beta}, \overbrace{2, 2, \ldots, 2}^{\gamma}, \overbrace{1, 1, \ldots, 1}^{\delta})$$

We certainly have $n = 1 + 4\alpha + 3\beta + 2\gamma + \delta$. Let

$$n = 1 + 4s + t$$

where $0 \leq t < 4$, and

$$M = (\overbrace{4, 4, \ldots, 4}^{s}, t)$$

then $C(M)$ is the 4-claw of size n. Certainly $M \geq L$. Since $C(L)$ is in T, by Lemma 1, we get that $C(M)$ is in T. It follows that $C(M) \in U(n)$ since T is arbitrary, i.e, the 4-claw of size n is in $U(n)$. Thus we finished the proof of Theorem 3 and Theorem 2 follows as well.

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